

MULTIDIMENSIONAL PROBABILISTIC REARRANGEMENT INVARIANT SPACES: A NEW APPROACH.

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Abstract.

We introduce and investigate in this paper a new convenient method of introduction of a norm in the multidimensional rearrangement probability invariant space.

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Martingale and martingale differences, exponential and moment estimations, rearrangement invariant, moment and Grand Lebesgue-Riesz spaces of random variables, multidimensional rearrangement invariant (m.d.r.i.) spaces, fundamental function, tail of distribution, polar, extremal points, random processes and fields, slowly and regular varying functions, sub-gaussian and pre-gaussian random variables, vectors and processes (fields), entropy and entropy integral, Young-Fenchel or Legendre transform.

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1 Introduction. Notations.

Let (Ω, F, \mathbf{P}) be a probability space, $\Omega = \{\omega\}$, $(X, \|\cdot\|_X)$ be any rearrangement (on the other terms, symmetrical) Banach function space over (Ω, F, \mathbf{P}) in the terminology of the classical book [3], chapters 1,2; see also [20]; $d = 2, 3, 4, \dots$,

$$S(d) = \left\{ b, b \in R^d, |b|_2 := \left[\sum_{j=1}^d b^2(j) \right]^{1/2} \leq 1 \right\}$$

be the Euclidean unit ball with the center at the origin, B be any separated complete closed subset of the set $S(d)$.

The last imply that for arbitrary $x \in R^d$ there exist two vectors $b_1, b_2 \in B$ for which $(b_1, x) \neq (b_2, x)$.

Definition 1. A Banach function space $(X^{(d)}, B)$ with the norm $\|\cdot\|_{(X^{(d)}, B)}$ consists by definition on all the d -dimensional random vectors (r.v.) $\xi = \vec{\xi} = \{\xi(1), \xi(2), \dots, \xi(d)\}$ with finite norm

$$\|\vec{\xi}\|_{(X^{(d)}, B)} = \|\xi\|_{(X^{(d)}, B)} \stackrel{def}{=} \sup_{b \in B} \|(\xi, b)\|_X \quad (1.1)$$

is called *multidimensional rearrangement invariant (m.d.r.i.) space based on the space X (Khintchin's version) and the set B* .

Here and furthermore (b_1, b_2) denotes as customary the inner (scalar) product of two d - dimensional vectors b_1, b_2 :

$$(b_1, b_2) = \sum_{j=1}^d b_1(j)b_2(j), \quad b_1 = \{b_1(j)\}, \quad b_2 = \{b_2(j)\}, \quad j = 1, 2, \dots, d.$$

Obviously, instead the whole set B in (1.1) it may be stand the set of extremal points one.

We will write for brevity in the case $B = S(d)$

$$\|\vec{\xi}\|(X^{(d)}, S(d)) = \|\xi\|(X^{(d)}, S(d)) = \|\xi\|X^{(d)}. \quad (1.2)$$

Evidently, the space $(X^{(d)}, B)$ is rearrangement invariant in the ordinary sense: the norm $\|\vec{\xi}\|(X^{(d)}, B)$ dependent only on the distribution of the random vector $\xi = \vec{\xi}$:

$$\mu_\xi(A) = \mathbf{P}(\xi \in A),$$

where A is arbitrary Borel set in the whole space R^d .

Another notations: the $L(p) = L_p = L_p(\Omega)$ is the classical Lebesgue-Riesz space consisting on all the random variables $\{\eta\}$ defined on the source probability space with finite norm

$$|\eta|_p \stackrel{def}{=} [\mathbf{E}|\eta|^p]^{1/p}, \quad p \geq 1.$$

For instance, the r.i. space $(X, \|\cdot\|_X)$ may be the classical $L_p(\Omega)$ space, Lorentz, Marzinkiewicz, Orlicz and so one spaces. We recall now for readers convenience briefly the definitions and simple properties of some new r.i. spaces, namely, Grand Lebesgue spaces and Banach spaces $\Phi(\phi)$ of random variables with exponentially decreasing tails of distribution (pre-gaussian and sub-gaussian spaces).

For $a = \text{const} > 1$ let $\psi = \psi(p)$, $p \in [1, a)$ be a continuous positive function such that there exists a limits (finite or not) $\psi(a - 0)$ with conditions $\inf_{p \in (1, a)} \psi(p) > 0$. We will denote the set of all these functions as $\Psi(a)$.

The Grand Lebesgue Space (in notation GLS) $G(\psi; a) = G(\psi)$ is the space of all measurable functions (random variables) $\eta : \Omega \rightarrow R$ endowed with the norm

$$\|\eta\|G(\psi) \stackrel{def}{=} \sup_{p \in (a, b)} \left[\frac{|\eta|_p}{\psi(p)} \right]$$

if it is finite.

These spaces are investigated, e.g. in the articles [6], [8], [9], [10], [11], [14], [15], [19], [22], [24], [25], [26], [27] etc. Notice that in the articles [9], [10], [11], [14], [15], [22], [25], [26] it was considered more general case when the measure \mathbf{P} may be unbounded.

For instance, in the case when $a < \infty, \beta = \text{const} > 0$ and

$$\psi(p) = \psi(a, \beta; p) = (p - a)^{-\beta}, 1 \leq p < a;$$

we will denote the correspondent $G(\psi)$ space by $G(a, \beta)$; it is not trivial, non-reflexive, non-separable etc.

Note that by virtue of Lyapunov inequality in the case $\beta = 0$, i.e. when the function $\psi(a; p)$ is bounded inside the *closed* interval $p \in [1, a]$ and is infinite in the exterior one, the space $G(a, \beta)$ is equivalent to the classical Lebesgue-Riesz space $L(a)$.

In the case when $a = \infty$ we need to take $\gamma = \text{const} > 0$ and define

$$\psi(p) = \psi(\gamma; p) = p^m, p \geq 1, m = \text{const} > 0.$$

We will denote for simplicity these spaces as $G(m)$.

We obtain a more general case considering the ψ function of a view

$$\psi(p) = p^m L(p), p \geq 1,$$

where the function $L = L(p)$ is positive continuous slowly varying as $p \rightarrow \infty$ function, so that the function $\psi(p)$ is regular varying as $p \rightarrow \infty$.

The following *sub-examples* are used in practice, see [22], [27]:

$$\psi(p) = p^m \log^\gamma(p + 1) L(\log p).$$

Further, let $\phi = \phi(\lambda), \lambda \in (-\lambda_0, \lambda_0), \lambda_0 = \text{const} \in (0, \infty]$ be some even strong convex which takes positive values for positive arguments twice continuous differentiable function, such that

$$\phi(0) = 0, \phi''(0) \in (0, \infty), \lim_{\lambda \rightarrow \lambda_0} \phi(\lambda)/\lambda = \infty.$$

We denote the set of all these function as Φ ; $\Phi = \{\phi(\cdot)\}$.

We will say that the *centered* random variable (r.v) $\eta = \eta(\omega)$ belongs to the space $\Phi(\phi)$, if there exists some non-negative constant $\tau \geq 0$ such that

$$\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbf{E} \exp(\lambda \eta) \leq \exp[\phi(\lambda \tau)].$$

The minimal value τ satisfying this inequality is called a $\Phi(\phi)$ norm of the variable ξ , write

$$\|\eta\|_{\Phi(\phi)} = \inf\{\tau, \tau > 0 : \forall \lambda \Rightarrow \mathbf{E} \exp(\lambda \xi) \leq \exp(\phi(\lambda \tau))\}.$$

This spaces are very convenient for the investigation of the r.v. having a exponential decreasing tail of distribution, for instance, for investigation of the limit theorem, the exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous of random fields, study of Central Limit Theorem (CLT) in the Banach spaces etc.

The space $\Phi(\phi)$ with respect to the norm $\|\cdot\|_{\Phi(\phi)}$ and ordinary operations is a Banach space which is isomorphic to the subspace consisted on all the centered variables of Orlicz's space $(\Omega, F, \mathbf{P}), N(\cdot)$ with N – function

$$N(u) = \exp(\phi^*(u)) - 1, \quad \phi^*(u) = \sup_{\lambda} (\lambda u - \phi(\lambda)),$$

see [19], [24], chapter 1.

The transform $\phi \rightarrow \phi^*$ is called Young-Fenchel transform. The proof of considered assertion used the properties of saddle-point method and theorem of Fenchel-Moraux:

$$\phi^{**} = \phi.$$

The next facts about the $B(\phi)$ spaces are proved in [6], [19], [24], chapter 1, [25]:

$$1. \eta \in B(\phi) \Leftrightarrow \mathbf{E}\eta = 0, \text{ and } \exists C = \text{const} > 0,$$

$$T(\eta, x) \leq \exp(-\phi^*(Cx)), x \geq 0,$$

where $T(\eta, x)$ denotes in this article the *tail* of distribution of the r.v. η :

$$T(\eta, x) = \max(\mathbf{P}(\eta > x), \mathbf{P}(\eta < -x)), x \geq 0,$$

and this estimation is in general case asymptotically exact.

More exactly, if $\lambda_0 = \infty$, then the following implication holds:

$$\lim_{\lambda \rightarrow \infty} \phi^{-1}(\log \mathbf{E} \exp(\lambda \eta)) / \lambda = K \in (0, \infty)$$

if and only if

$$\lim_{x \rightarrow \infty} (\phi^*)^{-1}(|\log T(\eta, x)|) / x = 1/K.$$

Here and further $f^{-1}(\cdot)$ denotes the inverse function to the function f on the left-side half-line (C, ∞) .

2. We define $\psi(p) = p/\phi^{-1}(p)$, $p \geq 2$. Let us introduce a new norm (the so-called "moment norm") on the set of r.v. defined in our probability space by the following way: the space $G(\psi)$ consist by definition on all the centered r.v. with finite norm

$$\|\eta\|_{G(\psi)} \stackrel{\text{def}}{=} \sup_{p \geq 2} [\|\eta\|_p / \psi(p)].$$

It is proved in particular that the spaces $\Phi(\phi)$ and $G(\psi)$ coincides: $\Phi(\phi) = G(\psi)$ (set equality) and both the norm $\|\cdot\|_{\Phi(\phi)}$ and $\|\cdot\|$ are equivalent: $\exists C_1 = C_1(\phi), C_2 = C_2(\phi) = \text{const} \in (0, \infty), \forall \eta \in \Phi(\phi)$

$$\|\eta\|_{G(\psi)} \leq C_1 \|\eta\|_{B(\phi)} \leq C_2 \|\eta\|_{G(\psi)}.$$

3. This definition is correct still for the non-centered random variables η . If for some non-zero r.v. η we have $\|\eta\|_{G(\psi)} < \infty$, then for all positive values u

$$\mathbf{P}(|\eta| > u) \leq 2 \exp(-u/(C_3 \|\eta\|G(\psi))).$$

and conversely if a r.v. ξ satisfies the last inequality, then $\|\xi\|G(\psi) < \infty$.

Let $\eta : \Omega \rightarrow R$ be some r.v. such that

$$\eta \in \cup_{p>1} L(p).$$

We can then introduce the non-trivial function $\psi_\eta(p)$ as follows:

$$\psi_\eta(p) \stackrel{def}{=} |\eta|_p.$$

This choosing of the function $\psi_g(\cdot)$ will be called *natural choosing*.

Analogously, if the centered (zero mean) r.v. η satisfies the Kramer's condition

$$\exists \mu \in (0, \infty), T(\eta, x) \leq \exp(-\mu x), x \geq 0,$$

the function $\phi(\cdot) = \phi_\eta(\lambda)$ may be "constructive" introduced by the formula

$$\phi(\lambda) = \phi_0(\lambda) \stackrel{def}{=} \log \mathbf{E} \exp(\lambda \eta),$$

if obviously the centered r.v. η satisfies the Kramer's condition:

$$\exists \mu \in (0, \infty), T(\eta, x) \leq \exp(-\mu x), x \geq 0.$$

We will call also in this case the function $\phi(\lambda) = \phi_\eta(\lambda)$ a *natural* function for the r.v. η .

The letters $C, C_k, C_k(\cdot), k = 1, 2, \dots$ with or without subscript will denote a finite positive non essential constants, not necessarily the same at each appearance.

Our aim is investigation of properties of m.d.r.i. spaces: some inequalities, conjugate and associate spaces, multidimensional moment and tail inequalities, estimation of normed sums of independent random vectors or martingale differences random vectors, finding of sufficient conditions for continuity of vector random processes and fields etc.

Example 1.1. Let $\xi = \vec{\xi} = \{\xi(1), \xi(2), \dots, \xi(d)\}$ be a centered random vector with finite covariation matrix

$$R(\xi) = \{R_{i,j}(\xi)\}, R_{i,j}(\xi) = \text{cov}(\xi(i), \xi(j)) = \mathbf{E} \xi(i) \cdot \xi(j), i, j = 1, 2, \dots, d.$$

If we choose $X = L_2 = L_2(\Omega)$, then

$$\|\xi\|^2(L_2^{(d)}) = \max_{b \in S(d)} \mathbf{E}(\xi, b)^2 = \max_{b \in S(d)} \sum_{i,j=1,2,\dots,d} R_{i,j}(\xi) b(i) b(j) = \lambda_{\max}(R(\xi)),$$

where $\lambda_{\max}(R(\xi))$ denotes the maximal eigen value of the matrix $\lambda_{\max}(R(\xi))$. Therefore,

$$\|\xi\|(L_2^{(d)}) = [\lambda_{\max}(R(\xi))]^{1/2}.$$

Example 1.2. Let $\xi = \vec{\xi} = \{\xi(1), \xi(2), \dots, \xi(d)\}$ be a centered *Gaussian* random vector with covariation matrix $R(\xi)$ and $(X, \|\cdot\|X)$ be arbitrary r.i. space. As long as the random variable (ξ, b) has the mean zero Gaussian distribution with variance $\text{Var}((\xi, b)) = (R(\xi)b, b)$, we conclude

$$\|\xi\|(X^{(d)}) = [\lambda_{\max}(R(\xi))]^{1/2} \cdot \|\tau\|X,$$

where τ has a standard Gaussian distribution.

If $X = L_p(\Omega)$, then

$$\|\tau\|X = \|\tau\|L_p(\Omega) = \sqrt{2} \pi^{-1/(2p)} \Gamma^{1/p}((p+1)/2),$$

where $\Gamma(\cdot)$ is Gamma function.

Example 1.3. We can choose instead the $(X, \|\cdot\|X)$ many other r.i. spaces over our probability space (Ω, F, \mathbf{P}) , for instance, the Marzinkiewicz M spaces, Lorentz spaces L , Orlicz's space N with correspondent N -function $N = N(u)$ and so ones.

The detail investigation of these spaces see in the classical monographs [3], chapters 1,2; [20], chapters 1,2; see also [1].

Example 1.4. In the capacity of the space X it may be represented the space $G(\nu; r)$, which consist, by definition, on all the r.v. with finite norm

$$\|\xi\|G(\nu; r) \stackrel{def}{=} \sup_{p \in (2, r)} [\|\xi\|_p / \nu(p)], \quad \|\xi\|_p := \mathbf{E}^{1/p} |\xi|^p.$$

Here $r = \text{const} > 2$, $\nu(\cdot)$ is some continuous positive on the *semi-open* interval $[1, r)$ function such that

$$\inf_{p \in (2, r)} \nu(p) > 0, \quad \nu(p) = \infty, \quad p > r.$$

We will denote

$$\text{supp}(\nu) \stackrel{def}{=} \{p : \nu(p) < \infty\}.$$

Sub-example:

$$\nu(p) = (r - p)^{-\gamma} L(1/(r - p)), \quad r = \text{const} > 1, \quad 1 \leq p < r, \quad \gamma = \text{const} \geq 0.$$

where as before $L = L(u)$ is positive continuous slowly varying as $u \rightarrow \infty$ function. About applications of these spaces see [28].

Other examples see in the section 5.

The paper is organized as follows. In the section 2 are investigated simple properties of introduces spaces. In the section 3 we study conjugate and associate spaces. In the next section we obtain some multidimensional tail inequalities for random vectors belonging to these spaces.

In the fifth section we formulate and prove a moment inequalities for sums of random vectors. The next section contains some information about fundamental function for m.d.r.i. spaces. The 7th section is devoted to a particular but important case of these spaces, namely, the so-called multidimensional spaces of random vectors of subgaussian and pre-gaussian type. Multidimensional random fields are considered in the 8th section.

The last section contains some concluding remarks.

2 Simple properties of multidimensional rearrangement invariant (m.d.r.i.) spaces.

Proposition 2.1. Let $\xi = \vec{\xi} = \{\xi(1), \xi(2), \dots, \xi(d)\}$ be a r.v. from the space $X^{(d)}$, then

$$\max_{i=1,2,\dots,d} \|\xi(i)\|X \leq \|\vec{\xi}\|(X^{(d)}) \leq \sum_{i=1}^d \|\xi(i)\|X; \quad (2.1a)$$

$$C_1 \max_{i=1,2,\dots,d} \|\xi(i)\|X \leq \|\vec{\xi}\|(X^{(d)}, B) \leq C_2 \sum_{i=1}^d \|\xi(i)\|X. \quad (2.1b)$$

Proof. As long as $b \in S(d)$, $|b(i)| \leq 1$, therefore

$$\|\vec{\xi}\|X^{(d)} = \max_{b \in B} |(\xi, b)|X = \max_{b \in B} \left| \sum_{i=1}^d b(i)\xi(i) \right|X \leq \sum_{i=1}^d \|\xi(i)\|X.$$

We prove the right hand side of inequality (2.1a).

Further, we have choosing the vector b of a view $b = (0, 0, \dots, 1, 0, \dots, 0) =: e(k)$, where "1" stands on the place k , $k = 1, 2, \dots, d$:

$$\|\vec{\xi}\|X^{(d)} \geq \|\xi(k)\|X,$$

hence

$$\|\vec{\xi}\|X^{(d)} \geq \max_{k=1,2,\dots,d} \|\xi(k)\|X.$$

The assertion (2.1b) may be proved analogously.

As a little consequences:

Proposition 2.2.

A. If the space $(X, \|\cdot\|X)$ is separable, then $(X^{(d)}, B)$ is separable.

B. If the space $(X, \|\cdot\|X)$ is reflexive, then $(X^{(d)}, B)$ is reflexive, as well as.

Note that an inverse conclusion is obvious.

Proposition 2.3.

It is easy to formulate the criterions for convergence of the sequences in these spaces and compactness of the sets. Indeed, convergence of the sequences is equivalent to coordinate-wise convergence; for the compactness of the sets are true the classical features of Kolmogorov and Riesz.

3 Conjugate and associate spaces.

We denote the *conjugate, or dual* space to the space $(X, \|\cdot\|_X)$ as $(X^*, \|\cdot\|_{X^*})$ and *associate* space to the space $(X, \|\cdot\|_X)$ as $(X', \|\cdot\|_{X'})$.

By definition, arbitrary linear continuous functional on the space $X^{(d)}$, in the other words, element of associate space of a view

$$l_g(\xi) = \sum_{i=1}^d \int_{\Omega} \xi(i, \omega) g_i(\omega) \mathbf{P}(d\omega), \quad (3.1a)$$

$$\xi = \vec{\xi} = \{\xi(1), \xi(2), \dots, \xi(d)\} = \{\xi(1, \omega), \xi(2, \omega), \dots, \xi(d, \omega)\}$$

is said to be an element of associate space: $g = \vec{g} = (g_1, g_2, \dots, g_d) \in X'$.

Analogously, arbitrary element of conjugate space $X^{(d)*}$, i.e. $h = h(\xi)$ may be uniquely represented on the form

$$h(\xi) = \sum_{i=1}^d h_i(\xi(i)). \quad (3.1b)$$

It is easy to verify that if the space X has Absolutely Continuous Norm (ACN):

$$\forall \eta \in X \Rightarrow \lim_{\mathbf{P}(A) \rightarrow 0+} \int_A |\eta(\omega)| \mathbf{P}(d\omega) = 0.$$

then $X^{(d)'} = X^{(d)*}$ and any linear continuous functional on the space $(X^{(d)}, B)$ may be uniquely represented by the formula (3.1a).

Proposition 3.1.

A. The expression (3.1a) represented an element of associate space $X^{(d)}'$ iff $\forall i \ g_i \in X'$ and

$$\max_i \|g_i\|_{X'} \leq \|l_g\|_{X^{(d)'}} \leq \sum_{i=1}^d \|g_i\|_{X'}. \quad (3.2a)$$

B. The expression (3.1b) represented an element of conjugate (dual) space $X^{(d)*}$ iff $\forall i \ h_i \in X^*$ and

$$\max_i \|h_i\|_{X^*} \leq \|h\|_{X^{(d)*}} \leq \sum_{i=1}^d \|h_i\|_{X^*}. \quad (3.2b)$$

Proof is at the same as the proof of proposition 2.1 and may be omitted.

For instance, let $X = L_p(\Omega) = L(p)$, where $1 < p < \infty$; denote $q = p/(p-1)$. Then $X' = X^* = L_q(\Omega)$ and we deduce by virtue of proposition 3.1 that every

linear continuous functional on the space $(L(p))^{(d)}$ may be uniquely represented by the formula (3.1a), where $g_i \in L(q)$ and

$$\max_i \|g_i\| L(q) \leq \|l_g\| (L(p))^{(d),*} \leq \sum_{i=1}^d \|g_i\| L(q). \quad (3.3)$$

4 Multidimensional tail inequalities.

Let D be arbitrary central-symmetric convex closed bounded set with non-empty interior in the space R^d . We will denote by $D(u)$, where u is "great" numerical parameter: $u \geq 1$ its u -homothetic transformation:

$$D(u) = \{x, x \in R^d, x/u \in D\}. \quad (4.0)$$

We intend to obtain in this section the exponential exact as $u \rightarrow \infty$ estimation for the probability

$$P_{D,\xi}(u) \stackrel{def}{=} \mathbf{P}(\xi \notin D(u)) \quad (4.1)$$

under assumption that $\xi \in X^d$.

For instance, if D is a unit Euclidean ball in the space R^d , then

$$P_{D,\xi}(u) \stackrel{def}{=} \mathbf{P}(|\xi|_2 > u).$$

We will use the following fact, see, e.g. [33], chapter 4, section 1: as long as the space R^d is finite-dimensional, $D = D^{oo} = M^o$, where $M = D^o$ denotes the *polar* of the set D :

$$M = D^o = \cap_{x \in D} \{y, (x, y) \leq 1\}.$$

Let us denote by $H(\text{extr}(M), \epsilon) =: H_M(\epsilon)$ the entropy of the set $\text{extr}(M)$ relative to the classical Euclidean distance and set $N_M(\epsilon) = \exp(H_M(\epsilon))$.

Theorem 4.1. Suppose the random vector ξ belongs to the space $L_p^{(d)}$. If the following integral converges:

$$I(M, p) \stackrel{def}{=} \int_0^1 N_M^{1/p}(\epsilon) d\epsilon < \infty, \quad (4.2)$$

then

$$P_{D,\xi}(u) \leq C(I(M, p), d) u^{-p}, u \geq 1. \quad (4.3)$$

Proof. We conclude by virtue of definition of the set M

$$P_{D,\xi}(u) = \mathbf{P}\left(\sup_{t \in M} (\xi, t) > u\right). \quad (4.4)$$

Obviously, instead the set M in the equality (4.4) may be used the set $\text{extr}(M)$ of all *extremal points* of the set M :

$$P_{D,\xi}(u) = \mathbf{P} \left(\sup_{t \in \text{extr}(M)} (\xi, t) > u \right). \quad (4.5)$$

Let us consider the (separable, moreover, continuous) random field $\eta(t) = (\xi, t)$, where $t \in \text{extr}(M)$. We have by means of definition of $\|\cdot\|_{(L_p^{(d)}, B)}$ norm:

$$\sup_{t \in \text{extr}(M)} |\eta(t)|_p \leq \|\xi\|_{L_p^{(d)}} < \infty, \quad (4.6)$$

and analogously

$$\forall t, s \in \text{extr}(M) \Rightarrow |\eta(t) - \eta(s)|_p \leq \|\xi\|_{L_p^{(d)}} \cdot |t - s|_2. \quad (4.7)$$

The assertion of theorem 4.1 follows immediately from the main result of paper belonging to G.Pizier [30].

Remark 4.1. The case when

$$\sup_{t \in \text{extr}(M)} \|\eta(t)\|_{G^{(d)}} \psi < \infty$$

and the correspondent distance

$$\rho(t, s) = \|\eta(t) - \eta(s)\|_{G^{(d)}} \psi$$

may be investigated analogously. See detail description with constant estimates in [24], chapter 3, section 3.17.

Remark 4.2. We conclude as long as $\text{extr}(M) \subset \partial D$ that if the set D has a smooth boundary, for instance, of the class C^1 piece-wise,

$$N(\epsilon) \leq C \cdot \epsilon^{-(d-1)}, \quad d \geq 2.$$

Remark 4.3. Note that the condition (4.2) is satisfied for all the values $p \in (0, \infty)$ if for example the set $\text{extr}(M)$ is finite: $N_M(\epsilon) \leq \text{card}(\text{extr}(M))$.

This occurs, e.g., when the set M is (multidimensional, in general case) polytop.

Example 4.1. Let $D = B$ be the standard unit ball in the space R^d ; then $M = \text{extr}(M) = \partial B$ is unit sphere in this space, the entropy integral $I(M, p)$ (4.2) converges iff $p > d - 1$ and in this case

$$P_{B,\xi}(u) \leq C(I(M, p), d) u^{-p}, \quad u \geq 1. \quad (4.8)$$

Evidently, the estimate (4.8) is true for arbitrary convex bounded domain D such that

$$\sup_{x \in D} |x| \leq 1.$$

Remark 4.3. Assume that the condition (4.2) is satisfied for any *diapason* (a, b) of a values p ; here $d - 1 < a < b \leq \infty$; then

$$P_{D,\xi}(u) \leq \inf_{p \in (a,b)} \left[C(I(M, p), d) u^{-p} \right], u \geq 1. \quad (4.9)$$

When $b = \infty$, then it may be obtained from (4.9) the exponential decreasing as $u \rightarrow \infty$ estimate for the probability $P_{D,\xi}(u)$.

5 Moment inequalities for sums of random vectors.

We recall before formulating the main result some useful for us moment inequalities for the sums of centered martingale differences (m.d.) $\theta(i)$ relative some filtration $\{F(i)\}$:

$$F(0) = \{\emptyset, \Omega\}, F(i) \subset F(i+1) \subset F :$$

$$\forall k = 0, 1, \dots, i-1 \Rightarrow$$

$$\mathbf{E}\theta(i)/F(k) = 0; \mathbf{E}\theta(i)/F(i) = \xi(i) \pmod{\mathbf{P}},$$

and for the independent r.v., [26]. Namely, let $\{\theta(i)\}$ be a sequence of centered martingale differences relative any filtration; then

$$\sup_n \sup_{b \in S(1)} \left| \sum_{i=1}^n b(i) \theta(i) \right|_p \leq K_M(p) \sup_i |\theta(i)|_p, \quad (5.0)$$

where for the *optimal value* of the constant $K_M = K_M(p)$ there holds the inequality

$$K_M(p) \leq p \sqrt{2}, \quad p \geq 2.$$

Note that the upper bound in (5.0)

$$K_I(p) \leq 0.87p / \log p, \quad p \geq 2$$

is true for the independent centered r.v. $\{\theta(i)\}$, see also [26].

Applying the inequality (5.0) for the value $n = d$, we obtain the following result.

Proposition 5.1.

A. Suppose the coordinates of the vector $\xi = \vec{\xi} = \{\xi(1), \xi(2), \dots, \xi(d)\}$ are centered martingale differences. Then

$$\|\vec{\xi}\|(L(p)^{(d)}) \leq K_M(p) \max_i |\xi(i)|_p. \quad (5.1a)$$

B. Suppose the coordinates of the vector $\vec{\xi} = \vec{\xi} = \{\xi(1), \xi(2), \dots, \xi(d)\}$ are centered independent r.v. Then

$$\|\vec{\xi}\|(L(p)^{(d)}) \leq K_I(p) \max_i |\xi(i)|_p. \quad (5.1b)$$

Let us denote for any function $\psi(\cdot) \in G\Psi$

$$\psi_K(p) = K_M(p)\psi(p), \quad \psi_I(p) = K_I(p)\psi(p).$$

A small consequence of a proposition 5.1:

Proposition 5.2.

A. Suppose the coordinates of the vector $\vec{\xi} = \xi = \{\xi(1), \xi(2), \dots, \xi(d)\}$ are centered martingale differences. Then

$$\|\vec{\xi}\|(G\psi_M^{(d)}) \leq \max_i \|\xi(i)\| G\psi. \quad (5.2a)$$

B. Suppose the coordinates of the vector $\vec{\xi} = \vec{\xi} = \{\xi(1), \xi(2), \dots, \xi(d)\}$ are centered independent r.v. Then

$$\|\vec{\xi}\|(G\psi_I^{(d)}) \leq \max_i \|\xi(i)\| G\psi. \quad (5.2b)$$

We intend now to generalize the famous Rozenthal's inequality on the multidimensional case. Let again $\psi \in \Psi$ and let $\eta = \eta(1)$ be a centered random vector from the space $G\psi^{(d)}$; let $\eta(2), \eta(3), \dots, \eta(n)$ be independent copies η . We denote

$$\zeta(n) = n^{-1/2} \sum_{j=1}^n \eta(j).$$

Proposition 5.3.

$$\sup_n \|\zeta(n)\|(G\psi_I^{(d)}) \leq \|\eta\|(G\psi^{(d)}). \quad (5.3)$$

Proof follows immediately from the classical Rozenthal's inequality, see, e.g. [17], [18], [26], [?], [32], [39]. Namely, let b be arbitrary deterministic vector from the set $S(d)$. We use the Rozenthal's inequality for the one-dimensional mean zero independent r.v. $\nu(j) = (\eta(j), b)$:

$$\sup_n |n^{-1/2} \sum_{j=1}^n \nu(j)|_p \leq K_I(p) \cdot |\nu|_p. \quad (5.4)$$

The assertion (5.3) follows from (5.4) after dividing on the $\psi(p)$ and taking maximum over b ; $b \in S(d)$ and p ; $\psi(p) \in (0, \infty)$.

Note that the estimate of a view

$$\sup_n \|\zeta(n)\|(G\psi_M^{(d)}) \leq \|\eta\|(G\psi^{(d)}) \quad (5.5)$$

is true for the sequence of (centered) martingale differences $\{\eta(j)\}, j = 1, 2, \dots, n$.

6 Fundamental function for m.d.r.i. spaces.

Recall that the fundamental function $\chi_X = \chi_X(\delta)$, $\delta \in (0, \mu(\Omega))$ for the r.i. space with measure $\mu(\cdot)$ $(X, \|\cdot\|_X)$ is defined by the formula

$$\chi_X(\delta) = \sup_{A, \mathbf{P}(A) \leq \delta} \|I(A)\|_X, \quad (6.0)$$

where as ordinary $I(A)$ is the indicator function of the event A .

We intend to generalize this definition in the multidimensional case of the space $X^{(d)}$. Namely, let $\vec{A} = \{A(i)\}, i = 1, 2, \dots, d$ be a *family* of measurable subsets of the whole space Ω and define the vector-function

$$I(\vec{A}) = \{I(A(1)), I(A(2)), \dots, I(A(d))\}.$$

The fundamental function of the m.d.r.i. space $X^{(d)}$ $\chi_{X^{(d)}}(\delta_1, \delta_2, \dots, \delta_d)$, $\delta_i \in [0, 1]$ may be defined as follows:

$$\begin{aligned} \chi_{X^{(d)}}(\delta_1, \delta_2, \dots, \delta_d) &= \sup_{A(i): \mathbf{P}(A(i)) \leq \delta_i} \|I(\vec{A})\|_{X^{(d)}} = \\ &= \sup_{A(i): \mathbf{P}(A(i)) \leq \delta_i} \sup_{b \in S(d)} \left\| \sum_{i=1}^d b(i) I(A(i)) \right\|_X. \end{aligned} \quad (6.1.)$$

Proposition 6.1.

$$\max_i \chi_X(\delta_i) \leq \chi_{X^{(d)}}(\delta_1, \delta_2, \dots, \delta_d) \leq \sum_{i=1}^d \chi_X(\delta_i). \quad (6.2)$$

Remark 6.1. Note that the lower bound in the bilateral inequality (6.2) is attained, for instance, when $X = L_p(\Omega)$, $p \geq 2$ and when the sets $A(i)$ are disjoint.

7 Multidimensional spaces of subgaussian and pre-gaussian type.

We consider in this section the case when the space $(X, \|\cdot\|_X)$ coincides with the space $\Phi(\phi)$ for some $\phi \in \Phi$.

Let the random vector $\vec{\xi} = \xi$ belongs to the space $\Phi^{(d)}$. Recall that this imply that $\mathbf{E}\vec{\xi} = \mathbf{E}\xi = 0$ and $\forall \lambda \in R$

$$\sup_{b \in S(d)} \mathbf{E} \exp\{\lambda(\xi, b)\} \leq \exp\{\phi(\lambda \cdot \|\xi\|_{\Phi^{(d)}})\},$$

or equally

$$\forall \mu \in R^d \Rightarrow \mathbf{E} \exp\{(\xi, \mu)\} \leq \exp\{\phi(|\mu|_2 \cdot \|\xi\|_{\Phi^{(d)}})\}. \quad (7.1)$$

Such a random vectors are called pre-gaussian.

We intend now to generalize the last definition. Let D be positive definite symmetrical constant matrix (linear operator) of a size $d \times d$. By definition, the (necessary mean zero) random vector $\xi = \vec{\xi}$ belongs to the space $\Phi_D(\phi)$, $\phi \in \Phi$, if there is a non-negative constant τ which dependent only on the distribution of the r.v. $\xi : \tau = \tau(\text{Law}(\xi))$ such that $\forall \lambda \in R$ and for arbitrary $b \in S(d)$

$$\mathbf{E} \exp(\lambda(\xi, b)) \leq \exp \phi \left(\lambda \tau \sqrt{(Db, b)} \right). \quad (7.2)$$

Definition 7.1.

The minimal value of the constant τ , $\tau \geq 0$ for which the inequality 7.2 is satisfied for all the prescribed values λ, b is said to be the ϕ, D norm of the random vector ξ :

$$\|\xi\|_{\phi, D} = \inf \{ \tau, \tau > 0, \forall \lambda \in R, \forall b \in S(d) \Rightarrow \mathbf{E} \exp(\lambda(\xi, b)) \leq \exp \phi \left(\lambda \tau \sqrt{(Db, b)} \right) \} \quad (7.3)$$

or equally

$$\|\xi\|_{\phi, D} = \inf_{\lambda \neq 0, b \in S(d)} \frac{\phi^{-1}(\log \mathbf{E} \exp(\lambda(\xi, b)))}{|\lambda| \|b\|_D}, \quad (7.4)$$

here and hereafter we denote

$$\|b\|_D = \sqrt{(Db, b)}.$$

We will denote the Banach space of all such a random vectors (i.e. with finite norm $\|\xi\|_{\phi, D}$) as $\Phi^{(d)}(\phi, D)$.

For instance, if a random vector η has Gaussian centered distribution with variation $R : \text{Law}(\xi) = N(0, R)$, then we can take $D = R$ and $\phi(\lambda) := \phi_0(\lambda) \stackrel{\text{def}}{=} 0.5\lambda^2$:

$$\|\eta\|_{\phi_0, R} = 1.$$

Definition 7.2.

The random vector ξ is said to be subgaussian relative the symmetric positive definite matrix D , if in the inequality (7.3) it can be taken $\phi(\lambda) = 0.5\lambda^2$:

$$\mathbf{E} \exp(\lambda(\xi, b)) \leq \exp \left(0.5\lambda^2 \tau^2 (Db, b) \right) \quad (7.5)$$

and the random vector ξ is said to be strong subgaussian, if in the inequality (7.5) it can be taken $\phi(\lambda) = 0.5\lambda^2$ and $D = R = \text{Var}(\xi)$:

$$\mathbf{E} \exp(\xi, \mu) \leq \exp(0.5(R\mu, \mu)). \quad (7.6)$$

This definitions belong to V.V.Buldygin and Yu.V.Kozatchenko, see [6], [7], where are described some applications. Another investigations and applications see in [29].

Definition 7.3.

Let ϕ be any function from the set Φ . The random vector ξ is said to be ϕ -strong subgaussian, if in the inequality (7.3) it can be taken $D = R = \text{Var}(\xi)$:

$$\mathbf{E} \exp(\lambda(\xi, b)) \leq \exp \phi \left(\lambda \tau \sqrt{(Rb, b)} \right). \quad (7.7)$$

Too more general definition. Let $\nu = \nu(\mu), \mu \in R^d$ be even: $\nu(-\mu) = \nu(\mu)$ strong convex which takes positive values for non-zero arguments twice continuous differentiable function, such that

$$\nu(0) = 0, \text{ grad } \nu(0) = 0, \lim_{r \rightarrow \infty} \min_{|\mu|_2 \geq r} \|\text{grad } \nu(\mu)\|_2 = \infty; \quad (A)$$

$$\inf_{\mu \in R^d} \lambda_{\min} \left[\frac{\partial^2 \nu}{\partial \mu_j \partial \mu_k} \right] > 0; \quad (B)$$

$$\nu(\mu(k) \times e(k)) = \nu(0, 0, \dots, 0, \mu(k), 0, 0, \dots, 0) \leq \nu(\vec{\mu}), \quad (C)$$

where

$$e(k) \stackrel{\text{def}}{=} (0, 0, \dots, 0, 1, 0, 0, \dots, 0),$$

and "1" stands on the place k .

We denote the set of all such a function as $\Phi^{(d)}$; $\Phi^{(d)} = \{\nu(\cdot)\}$.

For instance, the conditions (A), (B) and (C) are satisfied for non-degenerate centered (multidimensional) Gaussian distribution.

Definition 7.4.

We will say that the *centered* random vector $\eta = \eta(\omega)$ belongs to the space $\Phi^{(d)}(\nu)$, if there exists some non-negative constant $\tau \geq 0$ such that

$$\forall \mu \in R^d \Rightarrow \mathbf{E} \exp(\mu, \eta) \leq \exp[\nu(\mu \tau)].$$

The minimal value τ satisfying this inequality is called a $\Phi^{(d)}(\nu)$ norm of the variable η , write

$$\|\eta\|_{\Phi^{(d)}(\nu)} = \inf\{\tau, \tau > 0 : \forall \mu \in R^d \Rightarrow \mathbf{E} \exp(\mu \eta) \leq \exp(\nu(\mu \tau))\}.$$

Theorem 7.1. The set $\Phi^{(d)}(\nu)$ with ordinary operation equipped with correspondent norm $\|\cdot\|_{\Phi^{(d)}(\nu)}$ is (complete) r.i. Banach space over (Ω, F, \mathbf{P}) .

Proof is at the same as in one-dimensional case, see [19], [24], chapter 1; see also [2], and may be omitted.

Recall that the *multidimensional* Young-Fenchel, or Legendre transform $\nu^*(x)$, $x \in R^d$ of a function $\nu : R^d \rightarrow R$ is defined by the formula

$$\nu^*(x) = \sup_{\mu \in R^d} (x\mu - \nu(\mu)). \quad (7.9)$$

It is well known, [16], chapter 5, that $\nu^*(x)$ is convex function and if the function $\phi(\cdot)$ is convex and continuous on the its support, then

$$\nu^{**}(\mu) = \nu(\mu), \quad \mu : \nu(\mu) < \infty$$

(theorem of Fenchel-Morauux.)

The multidimensional tail function for the random vector ξ $T_\xi(x)$, $x \geq 0$, $x \in R^d$ may be defined as follows:

$$T_\xi(x) = \max_{\pm} \mathbf{P}(\pm\xi(1) \geq x_1, \pm\xi(2) \geq x_2, \dots, \pm\xi(d) \geq x_d),$$

where the exterior maximum is calculated over all combinations the signs \pm .

Theorem 7.2. The non-zero centered random vector ξ belongs to the space $\Phi^{(d)}(\nu, D)$ iff

$$T_\xi(\vec{x}) \leq \exp(-\nu^*(C\vec{x})) \quad (7.10)$$

(Tchernoff-Tchebychev's inequality).

Proof.

A. Let $\xi \in \Phi^{(d)}(\nu)$ and $\|\xi\| \Phi^{(d)}(\nu) = 1$. Denote as before also $x = \vec{x} = \{x(1), x(2), \dots, x(d)\}$, and assume without loss of generality that all the coordinates of the vector x are strictly positive.

As long as in the case $\max(x(i)) < 1$ the inequality (7.10) is evident, we consider further a possibility $\max(x(i)) \geq 1$.

We have by virtue of Tchebychev's inequality:

$$T_\xi(x) \leq \mathbf{E} \exp(\mu, \xi) / \exp(\mu, x) \leq \exp(-(\mu, x) - \nu(\mu)). \quad (7.11)$$

The assertion (7.10) is obtained from (7.11) after minimization over μ ; in the considered case it may be adopted $C = 1$.

B. Conversely, let the estimate (7.10) there holds with unit constant $C : C = 1$. Assume for definiteness $\vec{\mu} \geq 0$. As before, it is sufficient to consider only the case $\forall j = 1, 2, \dots, d \mu(j) \geq 1$.

We conclude after integration by parts:

$$\mathbf{E} \exp(\mu, \xi) \leq C + \prod_{i=1}^d \mu(i) \cdot \int_{R_+^d} \exp((\mu, x) - \nu^*(x)) dx. \quad (7.12)$$

We can estimate the last integral by means of saddle-point method, [12], chapter 2; see also [19]:

$$\begin{aligned} \mathbf{E} \exp(\mu \xi) &\leq C + \prod_{i=1}^d \mu(i) \cdot \sup_x \exp(C_2(\mu, x) - \nu^*(x)) = \\ &C + \prod_{i=1}^d \mu(i) \cdot \nu^{**}(C_3\mu) \leq \nu^{**}(C_4\mu) = \nu(C_4\mu); \end{aligned}$$

we used the theorem of Fenchel-Morauux.

This completes the proof of theorem 7.2.

Example 7.1.

A. Suppose for some $\tau = \text{const} > 0$, $\phi \in \Phi$ and for any strictly positive definite symmetrical matrix D of a size $d \times d$

$$\forall \mu \in R^d \Rightarrow \mathbf{E} \exp(\xi, \mu) \leq \exp(\phi(\tau \cdot |\mu|_D)),$$

then

$$T_\xi(x) \leq \exp(-\phi^*(x \cdot |\mu|_{D^{-1}}/\tau)).$$

B. Conversely, if $\mathbf{E}\xi = 0$ and for some constant $\tau > 0$

$$T_\xi(x) \leq \exp(-\phi^*(x \cdot |\mu|_{D^{-1}}/\tau)),$$

then there is a positive finite constant $C = C(\phi)$ such that

$$\forall \mu \in R^d \Rightarrow \mathbf{E} \exp(\xi, \mu) \leq \exp(\phi(C\tau \cdot |\mu|_D)).$$

In order to show the precision of result of theorem 7.2, let us consider the following

Example 7.2. Let $(\xi(1), \xi(2))$ be two-dimensional mean zero Gaussian distributed normed random vector such that

$$\mathbf{E}\xi^2(1) = \mathbf{E}\xi^2(2) = 1, \quad \mathbf{E}\xi(1)\xi(2) =: \rho \in (-1, 1).$$

It follows from the theorem 7.2 that at $x_1 > 0, x_2 > 0$

$$\mathbf{P}(\xi(1) > x_1, \xi(2) > x_2) \leq \exp\left(-0.5(1 - \rho^2)^{-1}(x_1^2 - 2\rho x_1 x_2 + x_2^2)\right),$$

but as $x_1 \rightarrow \infty, x_2 \rightarrow \infty$ independently inside the fixed angle

$$\arctan \rho + \varepsilon_0 \leq \frac{x_2}{x_1} \leq \arctan 1/\rho - \varepsilon_0,$$

$$\varepsilon_0 = \text{const} \leq 0.25(\arctan 1/\rho - \arctan \rho) :$$

$$\mathbf{P}(\xi(1) > x_1, \xi(2) > x_2) \sim C_{1,2}(\rho) x_1^{-1} x_2^{-1} \exp\left(-0.5(1 - \rho^2)^{-1}(x_1^2 - 2\rho x_1 x_2 + x_2^2)\right).$$

We continue the investigation of these spaces. Let the random vector $\vec{\xi} = \xi$ belongs to the space $\Phi^{(d)}(\nu)$ and let $k = 1, 2, \dots, d$. We denote

$$\nu_k(z) = \nu(z \times e(k)), \quad z \in R,$$

so that

$$\mathbf{E} \exp(z\xi(k)) \leq \exp(\nu_k(z));$$

$$\psi_k(p) := \frac{p}{\nu_k^{-1}(p)}.$$

Theorem 7.3. Suppose the function $\nu(\cdot)$ satisfies the conditions (A), (B) and (C).

On the (sub-)space of all *centered* random vectors $\{\xi = \vec{\xi}\}$ the norm $||\xi||\Phi^{(d)}(\nu)$ is equivalent to the other, so-called "moment norm":

$$|||\xi|||G\vec{\nu} \stackrel{def}{=} \sum_{k=1}^d \sup_{p \geq 1} \frac{|\xi(k)|_p}{\psi_k(p)} : \quad (7.13)$$

$$C_1(\nu)||\xi||\Phi^{(d)}(\nu) \leq ||\xi||G\vec{\nu} \leq C_2(\nu)||\xi||\Phi^{(d)}(\nu). \quad (7.14)$$

Proof.

A. Let $||\xi||\Phi^{(d)}(\nu) = 1$. From the direct definition of the norm $||\xi||\Phi^{(d)}(\nu)$ it follows

$$\forall \mu \in R^d \Rightarrow \mathbf{E} \exp(\mu, \xi) \leq \exp[\nu(\mu)]. \quad (7.15)$$

We substitute into inequality the value $\mu = z \cdot e_k$, $z \in R$:

$$\mathbf{E} \exp(z\xi(k)) \leq \exp(\nu_k(z)). \quad (7.16)$$

The inequality (7.16) means that the one-dimensional r.v. $\xi(k)$ belongs to the space $\Phi(\nu_k)$ and has there unit norm.

From the theory of these spaces [19], [24], chapter 1 it follows

$$\sup_{p \geq 1} \frac{|\xi(k)|_p}{\psi_k(p)} =: Y(k) < \infty,$$

therefore

$$|||\xi|||G\vec{\nu} \leq \sum_{k=1}^d Y(k) < \infty,$$

since the last sum is finite.

B. Let now $\mathbf{E}\xi = 0$ and $|||\xi|||G\vec{\nu} < \infty$; without loss of generality we can and will suppose $|||\xi|||G\vec{\nu} = 1$.

It follows from the last inequality that

$$\sup_{p \geq 1} \frac{|\xi(k)|_p}{\psi_k(p)} \leq 1$$

and hence

$$||\xi(k)||\Phi(\nu_k) =: Z_k < \infty.$$

We conclude by virtue of condition (C) of our theorem that

$$||\xi(k) \cdot e(k)||\Phi^{(d)}(\nu) = ||\xi(k)||\Phi(\nu_k) = Z_k.$$

As long as $\xi = \sum_{k=1}^d \xi(k) \cdot e(k)$, we obtain using triangle inequality

$$\begin{aligned} \|\xi\| \Phi^{(d)}(\nu) &\leq \sum_{k=1}^d \|\xi(k) \times e(k)\| \Phi^{(d)}(\nu) = \\ \sum_{k=1}^d \|\xi(k)\| \Phi(\nu_k) &= \sum_{k=1}^d Z_k < \infty, \end{aligned}$$

Q.E.D.

Proposition 7.1. Suppose $\xi \in \Phi^{(d)}(\nu), \xi \neq 0$. Denote

$$\overline{\nu}(\mu) = \sup_{n=1,2,\dots} n \nu(\mu/\sqrt{n}).$$

Note that the function $\overline{\nu}(\mu)$ there exists and satisfies the conditions (A), (B) and (C) since there exists a limit

$$\lim_{n \rightarrow \infty} n \nu(\mu/\sqrt{n}) = 0.5 (R\mu, \mu), \quad R = \text{Var } \xi.$$

Let $\{\xi^{(j)}\}, j = 1, 2, \dots$ be independent copies of ξ . Let us denote

$$S(n) = n^{-1/2} \sum_{j=1}^n \xi^{(j)}.$$

We observe as in the one-dimensional case:

$$\sup_n \|S(n)\| \Phi^{(d)}(\overline{\nu}) \leq \|\xi\| \Phi^{(d)}(\mu). \quad (7.17)$$

As a consequence we obtain an exponential bounds for normed sums of independent centered random vectors.

Proposition 7.2. Let $\{\xi^{(j)}\}, j = 1, 2, \dots$ be independent copies of mean zero non-trivial random vector ξ belonging to the space $\Phi^{(d)}$. We have the following uniform tail estimate for the norming sum of the r.v. $\{\xi^{(j)}\}$:

$$\sup_n T_{S(n)}(\vec{x}) \leq \exp \left(-\overline{\nu}^* \left(\vec{x} / \|\xi\| \Phi^{(d)}(\nu) \right) \right), \quad \vec{x} > 0. \quad (7.18)$$

8 Multidimensional random fields (processes).

Let $Y = \{y\}$ be arbitrary set and let (in this section) $\xi = \vec{\xi} = \vec{\xi}(y)$ be separable random field (process) with values in the space R^d :

$$\vec{\xi}(y) = \{\xi(1, y), \xi(2, y), \dots, \xi(d, y)\}.$$

The aim of this section is estimate the joint distribution (more exactly, joint tail function) as $\vec{v} = \{v_1, v_2, \dots, v_d\} \rightarrow \infty \Leftrightarrow \min v_i \rightarrow \infty$ for the coordinate-wise maximums:

$$U(\vec{v}) \stackrel{def}{=} \mathbf{P} \left(\sup_{y \in Y} \xi(1, y) > v_1, \sup_{y \in Y} \xi(2, y) > v_2, \dots, \sup_{y \in Y} \xi(d, y) > v_d \right) \quad (8.0)$$

or for brevity

$$U(\vec{v}) = \mathbf{P} \left(\sup_{y \in Y} \vec{\xi}(y) > \vec{v} \right), \vec{v} = (v_1, v_2, \dots, v_d),$$

$$\sup_{y \in Y} \vec{\xi}(y) = \{ \sup_{y \in Y} \xi(1, y), \sup_{y \in Y} \xi(2, y), \dots, \sup_{y \in Y} \xi(d, y) \}.$$

We denote

$$\bar{\xi}(j) = \sup_{y \in Y} \xi(j, y),$$

so that

$$U(\vec{v}) \stackrel{def}{=} \mathbf{P} \left(\bar{\xi}(1) > v_1, \bar{\xi}(2) > v_2, \dots, \bar{\xi}(d) > v_d \right).$$

The one-dimensional case $d = 1$ with described applications is in detail investigated in [24], introduction and chapter 4.

Notice that in a particular case when $v_1 = v_2 = \dots = v_d = v$ the probability $U(\vec{v})$ has a form:

$$U(v, v, \dots, v) = \mathbf{P} \left(\min_{i=1,2,\dots,d} \sup_{y \in Y} \xi(i, y) > v \right),$$

i.e. the probability $U(\vec{v})$ is the tail distribution for *minimax* of the random field $\xi(i, y)$.

We assume that

$$\forall y \in Y \Rightarrow \vec{\xi}(y) \in \Phi^{(d)}(\nu)$$

and moreover

$$\sup_{y \in Y} \|\vec{\xi}(y)\| \Phi^{(d)}(\nu) = 1. \quad (8.1)$$

Let us introduce the following *natural* distance, more exactly, *bounded*: $d(y_1, y_2) \leq 2$ semi-distance on the set Y :

$$d(y_1, y_2) = \|\vec{\xi}(y_1) - \vec{\xi}(y_2)\| \Phi^{(d)}(\nu). \quad (8.2)$$

Denote as usually for any subset V , $V \subset Y$ the so-called *entropy* $H(V, d, \epsilon) = H(V, \epsilon)$ as a logarithm of a minimal quantity $N(V, d, \epsilon) = N(V, \epsilon) = N$ of a (closed) balls $S(V, t, \epsilon)$, $t \in V$:

$$S(V, t, \epsilon) \stackrel{def}{=} \{s, s \in V, d(s, t) \leq \epsilon\},$$

which cover the set V :

$$N = \min\{M : \exists\{t_i\}, i = 1, 2, \dots, M, t_i \in V, V \subset \cup_{i=1}^M S(V, t_i, \epsilon)\},$$

and we denote also for the values $\epsilon \in (0, 1)$

$$H(V, d, \epsilon) = \log N; S(t_0, \epsilon) \stackrel{def}{=} S(Y, t_0, \epsilon), H(\epsilon) = H(d, \epsilon) \stackrel{def}{=} H(Y, d, \epsilon).$$

Here t_0 is the so-called "center" of the set Y relative the distance d , i.e. the point for which

$$\sup_{t \in Y} d(t_0, t) \leq 1.$$

We can and will assume the completeness of the space Y relative the distance d ; the existence of the center t_0 it follows from the Egoroff's theorem.

It follows from Hausdorff's theorem that $\forall \epsilon > 0 \Rightarrow H(Y, d, \epsilon) < \infty$ if and only if the metric space (Y, d) is pre-compact set, i.e. is the bounded set with compact closure.

Let p be arbitrary number from the interval $(0, 1/2)$. Define a function

$$w(p) = (1 - p) \sum_{n=1}^{\infty} p^{n-1} H(p^n). \quad (8.3)$$

We assume that

$$\exists p_0 \in (0, 1/2) \forall p \in (0, p_0) \Rightarrow w(p) < \infty, \quad (8.4)$$

(The so-called "entropy condition").

The condition (8.4) is satisfied if for example

$$H(\epsilon) \leq C + \kappa |\log \epsilon|; \quad (8.5)$$

in this case

$$w(p) \leq C + \frac{\kappa |\log p|}{1 - p}.$$

The minimal value κ for which the inequality (8.5) holds (if there exists) is said to be *entropy dimension* the set Y relative the distance d , write:

$$\kappa = \text{edim}(Y; d).$$

For instance, if Y is closed bounded subset of whole space R^m with non-empty interior and for some positive finite constant C_1, C_2 ; $0 < C_1 \leq C_2 < \infty$

$$C_1 |y_1 - y_2|^\alpha \leq d(y_1, y_2) \leq C_2 |y_1 - y_2|^\alpha, \alpha = \text{const} \in (0, 1],$$

where $|z|$ is ordinary Euclidean norm, then $\kappa = \text{edim}(Y; d) = d/\alpha$.

Theorem 8.1. Suppose the function $\nu(\cdot)$ satisfies the conditions (A), (B), (C). Let also the entropy condition (8.4) be satisfied. Then the vector random field $\vec{\xi}(y)$ is d -continuous with probability one:

$$\mathbf{P}(\vec{\xi}(\cdot) \in C(Y, d) = 1)$$

and moreover

$$U(v) \leq \inf_{p \in (0, p_0)} \exp[w(p) - \nu^*((v(1-p)))]. \quad (8.6)$$

Proof. The d -continuity of r.f. $\xi(\cdot)$ follows immediately from the main result of the paper [29]; it remains to prove the estimate (8.6).

We denote as $S(\epsilon)$ the minimal ϵ -net of the set Y relative the distance d : $\text{card}(S(\epsilon)) = N(\epsilon)$; not necessary to be unique, but non-random. By definition, $S(1) = \{t_0\}$ and $S_n = S(p^n)$ for arbitrary values p inside the interval $(0, p_0)$.

We define for any $y \in Y$ and $n = 0, 1, 2, \dots$ the non-random functions $\theta_n(\cdot)$, (projection into the set S_n) also not necessary to be unique, as follows: $\theta_0(y) = y_0$ and

$$\theta_n(y) = y_j, \quad y_j \in S_n, \quad d(y, y_j) \leq p^n, \quad n = 1, 2, \dots \quad (8.7)$$

The set

$$S = \cup_{n=0}^{\infty} S(p^n) \quad (8.8)$$

is enumerate dense subset of the space (Y, d) ; we can choose the set S as the set of separability for the random field $\vec{\xi}(t)$. We have:

$$\sup_{y \in Y} \xi(y) = \lim_{n \rightarrow \infty} \max_{y \in S_n} \xi(y).$$

Further, when $n \geq 1$

$$\max_{y \in S_n} \xi(y) \leq \max_{y \in S_n} (\xi(y) - \xi(\theta_{n-1}y)) + \max_{y \in S_{n-1}} \xi(y),$$

therefore

$$\sup_{y \in Y} \xi(y) \leq \sum_{n=0}^{\infty} \eta_n, \quad (8.9)$$

where

$$\eta_0 = \xi(y_0), \quad \eta_n = \max_{y \in S_n} (\xi(y) - \xi(\theta_{n-1}y)), \quad n = 1, 2, \dots$$

We have for positive values $\mu = \vec{\mu}$: $\mathbf{E} \exp(\mu \eta_0) \leq \exp(\nu(\mu))$, $\mathbf{E} \exp(\mu \eta_n) \leq$

$$\sum_{y \in S_n} \mathbf{E} \exp((\mu, \xi(y) - \xi(\theta_{n-1}y))) \leq \sum_{y \in S_n} \exp(\nu(\mu \cdot p^{n-1})) \leq$$

$$N(p^n) \exp(\nu(\mu \cdot p^{n-1})) = \exp(H(p^n) + \nu(\mu \cdot p^{n-1})).$$

As long as

$$\mathbf{E} \exp(\mu \cdot \sup_{y \in Y} \xi(y)) \leq \mathbf{E} \exp(\mu \cdot \sum_n \eta_n) = \mathbf{E} \prod_{n=0}^{\infty} \exp(\mu \cdot \eta_n).$$

We obtain using Hölder's inequality in which we choose $1/r(n) = p^n(1-p)$, so that $r(n) > 1$, $\sum_{n=0}^{\infty} 1/r(n) = 1$:

$$\begin{aligned} \mathbf{E} \exp(\mu \cdot \sup_{y \in Y} \xi(y)) &\leq \prod_{n=0}^{\infty} [\mathbf{E} \exp(\mu \cdot r(n) \cdot \eta_n)]^{1/r(n)} \leq \\ &\prod_{n=0}^{\infty} [\exp(H(p^n) + \nu(r(n) \cdot \mu \cdot p^n))]^{1/r(n)} \leq \\ &\exp \left\{ w(p) + \nu \left(\frac{\mu}{1-p} \right) \right\}. \end{aligned} \quad (8.10)$$

It remains to use Tchebychev's - Tchernoff's inequalities in order to obtain the assertion (8.6) of theorem 8.1 for arbitrary fixed value p and with consequent optimization over p .

Corollary 8.1. Let us denote

$$\pi(v) = \frac{C}{(\nabla \nu^*(v), v)}$$

and assume that $\pi(v)$ there exists and

$$\lim_{\min v(i) \rightarrow \infty} \pi(v) = 0.$$

We deduce choosing in (8.6) the value $p = \pi(v)$ for all sufficiently great values $v = \vec{v}$: $\pi(v) \leq p_0$:

$$U(v) \leq \exp \left[w \left(\frac{C_2}{\pi(v)} \right) - \nu^*(v) \right]. \quad (8.11)$$

Example 8.1. Suppose in addition to the conditions of theorem 8.1 the condition 8.5 holds. Then

$$U(v) \leq C(\kappa) \pi(v)^{-\kappa} \exp(-\nu^*(v)), \quad \min_i v_i > v_0. \quad (8.12)$$

Example 8.2. Let $\xi(y) = (\xi(1, y), \xi(2, y))$, $y \in Y$ be a two-dimensional centered Gaussian random field with the following covariation symmetrical positive definite matrix-function: $R = R(y_1, y_2) = \{R_{i,j}(y_1, y_2)\}, i, j = \{1; 2\}$,

$$\text{cov}(\xi(i, y_1), \xi(j, y_2)) = \mathbf{E} \xi(i, y_1) \cdot \xi(j, y_2) = R_{i,j}(y_1, y_2).$$

Suppose that the matrix $R(y_1, y_2)$ for all the values $y_1, y_2 \in Y$ is less then the following constant matrix $E^{(\rho)} = \{E_{i,j}^{(\rho)}\}, i, j = \{1; 2\}$: $R(y_1, y_2) \ll E^{(\rho)}$ with entries

$$E_{1,1}^{(\rho)} = E_{2,2}^{(\rho)} = 1, \quad E_{1,2}^{(\rho)} = E_{2,1}^{(\rho)} = \rho,$$

where $\rho = \text{const} \in (-1, 1)$.

Recall that the inequality of a view $A \ll B$ between two square symmetrical matrices A and B with equal size $l \times l$ is understood as usually: $A \ll B$ iff

$$\forall x \in R^l \Rightarrow (Ax, x) \leq (Bx, x).$$

Assume also the distance d satisfies the condition (8.5). It follows from the inequality (8.12) that in the considered case

$$U(v) \leq C(\kappa) (v_1^2 - 2\rho v_1 v_2 + v_2^2)^\kappa \exp\left(-0.5(1 - \rho^2)^{-1}(v_1^2 - 2\rho v_1 v_2 + v_2^2)\right), \quad (8.13)$$

when $\min_i v_i \geq 1$.

Analogous result may be formulated in the multidimensional case.

Remark 8.1. More fine result may be obtained by means of the so-called generic chaining method, see [21], [35], [36], [37], [38].

9 Concluding remarks

Embedding theorems.

For the considered in this article m.d.r.i. spaces may be obtained embedding theorems alike in the one-dimensional case, see [24], chapter 1, section 1.17.

Arbitrary measure.

It may be considered also the case when the measure μ is unbounded, but sigma-finite. The one-dimensional case oh these spaces are investigated, e.g. in [25], [29].

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